Reduction of a Finite Element Parametric Model using Adaptive POD Methods

S. Clénet[1,3], T. Henneron[2], N. Ida[3]

[1]L2EP, Arts et Métiers ParisTech, 59046 Lille, France.
[2]L2EP, Université Lille 1, 59655 Villeneuve d'Ascq, France.
[3]Department of Electrical and Computer Engineering, University of Akron, USA

Model Order Reduction (MOR) methods enable reduction of the computation time when dealing with parametrized numerical models. Among these methods, the Proper Orthogonal Decomposition (POD) method seems to be a good candidate because of its simplicity and its accuracy. However, the accuracy strongly depends on the choice the parameter set chosen to construct the reduced basis. In this communication, we propose three different procedures for an adaptive construction of the parameter set. The accuracy of the three methods is compared on a 2D Finite Element model example in magnetostatics.

***Index Terms*— Finite Element Method (FEM), Model Order Reduction (MOR), Proper Orthogonal Decomposition (POD), Error estimation**

I. Introduction

To design an electromagnetic device or to quantify the impact of uncertainties, parameterized models are needed.

In order to obtain good accuracy, numerical models based on the Finite Element (FE) method are often used. The issue is then the computation time which can be very long especially if the number of parameters is high, more than about a dozen. Recently, Model Order Reduction (MOR) methods, like the Proper Orthogonal Decomposition (POD) method or the Reduced Basis (RB) method, have been applied in computational electromagnetics for uncertainty quantification or design [1,2]. An approximation of the full parametrized model (the original FE model) is then sought in a space spanned by a reduced basis, which enables one to reduce the number of degrees of freedom [3]. The equation system is much smaller than the full model. However, the accuracy of the reduced model is strongly related to the choice of the reduced basis which is obtained from the solutions of the full problem for particular parameter values.

In this communication, we propose to compare three iterative procedures to determine the reduced basis based on different error indicators. These approaches are compared in terms of accuracy on an example, a magnetic holder whose geometry is defined by 11 parameters.

II. Reduction of a Parametric model Using Pod Method

The discretization of a parameterized magnetostatic problem using the FE method leads to the following linear system of equations:

$$\mathbf{S}(\mathbf{p})\,\mathbf{X}(\mathbf{p}) = \mathbf{F}(\mathbf{p}) \tag{1}$$

Where $\mathbf{p}=(p_1,..,p_M)$ are the M input parameters, $\mathbf{S}(\mathbf{p})$ is the N×N stiffness matrix, $\mathbf{F}(\mathbf{p})$ is the N×1 source vector and N is the number of degrees of freedom. The solution $\mathbf{X}(\mathbf{p})$ enables one to determine the field distribution and also the quantities of interest which are usually either linear functions of $\mathbf{X}(\mathbf{p})$ (flux) or quadratic functions of $\mathbf{X}(\mathbf{p})$ (energy or force). If we denote Q as the quantity of interest which is a quadratic function of $\mathbf{X}(\mathbf{p})$, it can be written under the form:

$$Q=\mathbf{X}(\mathbf{p})^t\,\mathbf{D}(\mathbf{p})\,\mathbf{X}(\mathbf{p}) \tag{2}$$

With $\mathbf{D}(\mathbf{p})$ a matrix. If the model (1) and (2) is applied in a design process or for uncertainty quantification, the number of calls (solutions) can be very high especially if the parameter number M is significant due to the "curse of dimensionality". To decrease the computation time, one should either limit the number of calls or reduce the size of the linear system of equations (1). The POD method, detailed in the following, is one of the most popular MOR methods. Consider a set of Z parameters $(\mathbf{p_1},\ldots,\mathbf{p_Z})$ and the N×Z matrix $\boldsymbol{X}$ of the associated solutions $(\mathbf{X}(\mathbf{p_1}),\ldots,\mathbf{X}(\mathbf{p_Z}))$. We define the linear space K spanned by the vectors $(\mathbf{X}(\mathbf{p_1}),\ldots,\mathbf{X}(\mathbf{p_Z}))$ and the N×R matrix $\boldsymbol{\Psi}$ (R≤Z) of the vectors $(\boldsymbol{\Psi_1},..,\boldsymbol{\Psi_R})$, an orthogonal basis of the space K. The matrix $\boldsymbol{\Psi}$ can be obtained by a Singular Value Decomposition (SVD) of the matrix $\boldsymbol{X}$. The idea of the POD method is to seek an approximation of the solution of (1) in the space K, which means that $\mathbf{X}(\mathbf{p})$ is approximated by the following linear combination:

$$\mathbf{X}(\mathbf{p}) \approx \boldsymbol{\Psi}\,\mathbf{X_r}(\mathbf{p}) = \sum_{i=1}^{R} x_{ri}(\mathbf{p})\boldsymbol{\Psi_i} \tag{3}$$

The approximation has to satisfy (1), which is not generally possible because the equation system is overdetermined. By applying the Galerkin method, the vector $\mathbf{X_r}(\mathbf{p})$ has to satisfy an R×R linear system of equations:

$$\boldsymbol{\Psi}^t\mathbf{S}(\mathbf{p})\,\boldsymbol{\Psi}\,\mathbf{X_r}(\mathbf{p}) = \boldsymbol{\Psi}^t\mathbf{F}(\mathbf{p}) \tag{4}$$

The size of the system (4) is then equal to R, which is much lower than N, the size of the full system (1). The solution of the system is much faster reducing significantly the computation time. The accuracy of the method is closely related to the choice of the parameters $(\mathbf{p_1},\ldots,\mathbf{p_Z})$ used to determine the reduced basis. In the following, we propose different strategies to determine them adaptively.

III. Adaptive procedure

To determine adaptively the parameter set, an error

indicator is needed that provides information on the error between the solution $\mathbf{X}(\mathbf{p})$ of (1) and the solution $\mathbf{X_r}(\mathbf{p})$ of (4) for a given value of the input parameters $\mathbf{p}$. The idea is to obtain this information without solving (1), which is time consuming, that is to say without knowing $\mathbf{X}(\mathbf{p})$. Let us consider the step n of adaptive procedure to construct the reduced basis $\boldsymbol{\Psi}_n$. The reduced basis has been obtained by solving (1) for a parameter set $(\mathbf{p_1},\ldots,\mathbf{p_n})$. We select now a set of new parameters $(\mathbf{p}^1,..,\mathbf{p}^k)$. In our case, k is equal to the number of parameters M and the $\mathbf{p}^i$'s are determined from $\mathbf{p_n}$ by changing only the i^{th} component of $\mathbf{p_n}$. The reduced problem (4) is solved for the k parameters and we denote $\mathbf{X_r}(\mathbf{p}^i)$ the k solutions. Then, the parameter $\mathbf{p}^i$ leading to the highest error is selected to complete the parameter set. Three error indicators are proposed. The first error indicator $\varepsilon_1^{\,i}$ consists in calculating the residual of (1):

$$\varepsilon_1^{\,i}=\mathbf{R_i}^t\,\mathbf{R_i} \text{ with } \mathbf{R_i}=\mathbf{F}(\mathbf{p}^i)-\mathbf{S}(\mathbf{p}^i)\,\boldsymbol{\Psi}_n\,\mathbf{X_r}(\mathbf{p}^i) \qquad (5)$$

We can see that if $\mathbf{X_r}(\mathbf{p}^i)$ is equal to $\mathbf{X}(\mathbf{p}^i)$ then the indicator is equal to zero. We introduce a second error indicator $\varepsilon_2^{\,i}$ that is more oriented towards the calculation of the quantity of interest Q defined in (2) and which is given by:

$$\varepsilon_2^{\,i}=\mathbf{R_i}^t\,\mathbf{D}(\mathbf{p}^i)\,\mathbf{R_i} \qquad (6)$$

Finally, we introduce an indicator $\varepsilon_3^{\,i}$ based on the error estimator proposed in [4] which is given by:

$$\varepsilon_3^{\,i}=\mathbf{R_i}^t\,\mathbf{S}^{-1}(\mathbf{p_0})\,\mathbf{R_i} \qquad (7)$$

The parameter $\mathbf{p_0}$ is a "well chosen" value of $\mathbf{p}$. The calculation of the inverse of $\mathbf{S}(\mathbf{p_0})$ has to be done only once before the adaptive process. The next parameter value $\mathbf{p}_{n+1}$ is taken equal to $\mathbf{p}^i$ which corresponds to the highest value of $\varepsilon_j^{\,i}$, with j equal to 1,2 or 3 depending on the error indicator used. Now, the full problem (1) is solved for $\mathbf{p}=\mathbf{p_{n+1}}$. A new reduced basis $\boldsymbol{\Psi}_{n+1}$ is calculating from $\boldsymbol{X}_{n+1}=(\mathbf{X}(\mathbf{p_1}),\ldots,\mathbf{X}(\mathbf{p_n}),\mathbf{X}(\mathbf{p_{n+1}}))$. The process is repeated until the error indicator is sufficiently small.

IV. Example of Application

We consider a magnetic holder modelled by a 2D FE vector potential formulation. The geometry of the device is defined by 11 parameters, which are represented in Fig. 1. The quantity of interest is the force experienced by the mobile plate when the coil is not energized (due only to the permanent magnet). The force has been calculated using the Maxwell Stress Tensor and can be expressed under the form (2). We have fixed nominal values for the parameters p_i^{nom} and consider an interval of variation of $[0.1p_i^{nom},1.9p_i^{nom}]$ for each parameter. We have applied the previous adaptive process for the three error indicators. This leads to three different reduced basis and so three different reduced models (4) denoted RM1, RM2 and RM3 obtained from the error indicators (5), (6) and (7) respectively. In order to compare the accuracy of the reduced models, we have generated a sample of N parameter realizations $(\mathbf{p'_1},\ldots,\mathbf{p'_N})$ is generated using the Latin Hypercube Sampling technique. We solve the three reduced problems and the full problem (1) for the N realizations and calculate the forces $F_{RMj}(\mathbf{p'_i})$ and $F_{full}(\mathbf{p'_i})$ and the energies $E_{RMj}(\mathbf{p'_i})$ and $E_{full}(\mathbf{p'_i})$.

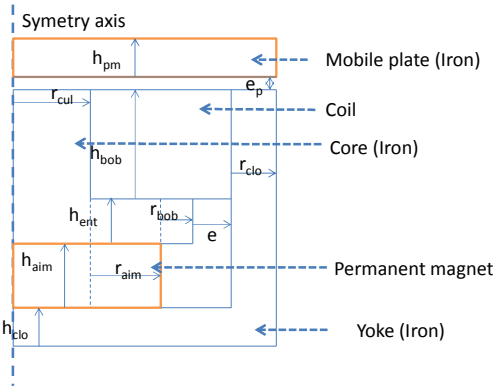

Fig.1. Half of the geometry of the magnetic holder and the definition of the parameters $(r_{cul},r_{aim},r_{bob},e,r_{clo},e_p,h_{clo},h_{aim},h_{ent},h_{bob},h_{pm})$

The errors in the L^2 and L^∞ sense are defined as:

$$\mathbf{err_j}^2=\frac{\sqrt{\sum_{i=1}^{N}\left(F_{RMj}(\mathbf{p_i})-F_{full}(\mathbf{p_i})\right)}}{\sum_{i=1}^{N}F_{full}(\mathbf{p_i})}$$

$$\mathbf{err_j}^\infty=\frac{\max_{1\le i\le N}\left|F_{RMj}(\mathbf{p_i})-F_{full}(\mathbf{p_i})\right|}{\sum_{i=1}^{N}F_{full}(\mathbf{p_i})} \qquad (8)$$

Table 1 shows the errors obtained for a reduced basis of size Z=10 and Z=30 with the three reduced models. The three error indicators leads to three different parameter sets $(\mathbf{p_1},\ldots,\mathbf{p_Z})$ and so to three different models. We can see clearly an error reduction when the size Z of the reduced basis increases. The error is of the same order in the L^2 sense for the three reduced models. However, we can notice that in the L^∞ sense, RM3 seems to be a little bit more accurate than the other two.

TABLE I: Estimation of the error L^2 and L^∞ on the force and the energy for Z=10 and Z=30

Error indicator	Z	L^2force	L^∞force	L^2energy	L^∞energy
RM1,err_1	10	8.6	47.4	9.4	68.38
RM1 err_1	30	2.1	13.6	2.5	21.7
RM2 err_2	10	6.1	48.8	8.6	69.8
RM2 err_2	30	3.0	27.6	4.0	43.8
RM3 err_3	10	6.8	29.0	4.7	20.2
RM3 err_3	30	2.6	8.12	2.58	12.9

Acknowledgement

This work was supported by the foundation Arts et Métiers

References

[1] M.A. Drissaoui, S. Lanteri, P. Leveque, F. Musy, L. Nicolas, R. Perrussel, D. Voyer, "A Stochastic Collocation Method Combined With a Reduced Basis Method to Compute Uncertainties in Numerical Dosimetry", *IEEE Trans. Mag.*, vol. 48(2), 2012.

[2] Y. Sato, F. Campelo. H. Igarashi1,Fast Shape Optimization of Antennas Using Model Order Reduction, CEFC2014, Aix les Bains, France

[3] D. Schmidthausler, S. Schops, M. Clemens, "Reduction of Linear Subdomains for Non-Linear Electro-Quasistatic Field Simulations", *IEEE Trans. Mag.*, vol. 49(5), 2013.

[4] H. Mac, S. Clénet, "A posteriori error estimation for stochastic static problems", *IEEE Trans. Mag.*, vol. 50(2), 2014